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ON (P, S)-RESIDUE SYSTEM MODULO *N*

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ABSTRACT

In this paper we generalize the notion of reduced residue system (mod n) using direct factor sets and regular divisor of n.

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1. INTRODUCTION

A non empty set P of positive integers is called a *direct factor set* if for n_1, n_2 with $\gcd(n_1,n_2)=1$ we have $n_1n_2 \in P \Leftrightarrow n_1 \in P \, and \, n_2 \in P$. A pair P and Q of direct factor sets is said to form a *conjugate pair* if every positive integer *n* can be written uniquely as $n = ab$, where $a \in P$ and $b \in Q$. For such a pair note that $P \cap Q = \{1\}$. As example of conjugate pairs of direct factor sets we have (i) $P = \{1\}$, $Q = N$ (set of all natural numbers) and (ii) $P =$ the set of all k-free integers (that is, the integers not divisible by the k^{th} power of any prime) and $Q =$ the set of all k^{th} power of positive integers. For any integer $n > 1$, S_n denotes a set of positive divisors of n. The elements of a complete residue system (mod n) such that $(a,n)_S \in P$ where $(a,n)_S$ denotes the greatest divisor of *a* in S_n, is called a (P,S)*-reduced residue system (mod n)* or simply a (P,S) *- system (mod n)*. A (P,S) - System (mod n) from the numbers 1,2,3,…,*n* will be called a *minimal* (P,S) *- system (mod n)*. In case $S_n = D_n$, the set of all positive divisors of *n,* we note that a (P,S) -system (mod*n*)is the P-reduced residue system (mod*n*) defined by Eckford Cohen[2].

The number of elements in a (P,S) -system (mod*n*) is denoted by $\varphi_{P,S}(n)$ and it is called the (P,S) *-totient function*. Further it may be observed that in the case $P = \{1\}$ the totient $\varphi_{P,S}(n)$ reduces to $\varphi_S(n)$, the S-analogue of the Euler totient function discussed by P. J. McCarthy[4] and others.

The purpose of this paper is study this function $\varphi_{P,S}(n)$.

2. PRELIMINARY RESULTS

For any integer $n > 1$, let S_n be a nonempty set of positive divisors of n. Let A be the class of all arithmetic functions. For $f, g \in A$ their *S-convolution,* $f \, \overline{S} \, g$, is defined by

$$\left(f\,\overline{S}\,g\right)(n)=\sum_{d\in S_n} f(d)g\left(\frac{n}{d}\right), \tag{2.1}$$

Where the sum is over all elements of d of S_n.

Observe that if $S_n = D_n$ (the set of all positive divisors of n) then $\left(f\,\overline{D}g\right)(n)=\left(f * g\right)(n)$, where $*$ is the classical Dirichlet convolution. Also if $S_n = U_n$ (the set of all unitary divisors of n (recall d is a unitary divisor of n if $d|n$ and $\gcd\left(d,\frac{n}{d}\right)=1$) we have $\left(f\,\overline{U}g\right)(n)=\left(f\circ g\right)(n)$, where $\circ$ is the unitary convolution studied by Eckford Cohen[3].

Introducing S-convolutions, Narkiewicz [5] has obtained a set of necessary and sufficient conditions on the set S_n, so that the following holds:

- $(A,+,\overline{s})$ is a commutative ring with unity (in which ε given by $\varepsilon(n)=1\,or\,0$ according as $n=1$ or $n>1$ is the unity)
- $f\,\overline{S}\,g$ is multiplicative whenever f and g are.
- The arithmetic function $u(n)=1$ for all n has inverse $\mu_s \in A$ relative to $\overline{s}$ (that is, $u\,\overline{s}\,\mu_s=\varepsilon$) and $\mu_s(n)=0$ or -1 when n is a prime power. μ_s is called the S-analogue of the Mobius function μ.

Such a S-convolution is called a *regular convolution.* Note that both Dirichlet convolution and Unitary convolution are regular.

V. Siva Rama Prasad and M. Ganeshwar Rao [6] have introduced a *generalized Mobius function* $\mu_{P,S}$ and it is defined by

$$\mu_{P,S}(n)=\sum_{d\in S_n\cap P}\mu_S\left(\frac{n}{d}\right) \tag{2.2}$$

Where μ_S is the S-analogue μ. Note that

$$\mu_{P,S}(n)=\left(\chi_P\overline{s}\mu_S\right) \tag{2.3}$$

Where χ_P is the characteristic function of P (that is, $\chi_P(n)=1$ or 0 according as $n\in P$ or $n\notin P$). In fact

$$\mu_{P,S}(n)=\sum_{d\in S_n\cap P}\mu_S\left(\frac{n}{d}\right)$$

$$=\sum_{d\in S_n\cap P}\chi_P(d)\mu_S\left(\frac{n}{d}\right)$$

$$=(\chi_P \bar{S} \mu_S)(n).$$

Also they have established a generalized inversion formula given below:

Let P, Q *be a conjugate pair of direct factor sets. Then for* $f, g \in A$

$$g(n)=\sum_{d\in S_n\cap Q} f\left(\frac{n}{d}\right) \Leftrightarrow f(n)=\sum_{d\in S_n} g(d)\mu_{P,S}\left(\frac{n}{d}\right) \quad \textbf{(2.4)}$$

Remark: In Case $S_n = D_n$, (2.4) gives the inversion formula proved by Eckford Cohen [2]. Also if $S_n = U_n$, $P =$ the set of all k-free integers and $Q =$ the set of all k^{th} power of positive integers the inversion formula due to Suryanarayana [7] is obtained from (2.4).

3. THE FUNCTION

$\varphi_{P,S}$

In all that follows P and Q form a conjugate pair of direct factor sets. Also $\bar{S}$ is a regular convolution on the class A of all arithmetic functions.

3.1. Theorem: Suppose $d \in S_n \cap Q$ and for each *d*, let X ranges over the elements of a (P, S)-system $\left(\mathrm{mod}\,\frac{n}{d}\right)$. Then the set dX forms a complete residue system $(\mathrm{mod}\, n)$.

Proof: For any fixed $d \in S_n \cap Q$, let $C_d = \{0 < a \le n, (a,n)_s = de, e \in P\}$. Then any a with $0 < a \le n$ lies exactly in one class C_d. Hence the union $\bigcup_{d\in S_n\cap Q} C_d$ contains all the integers 1,2,3,…,*n*. Also for a fixed $d \in S_n \cap Q$ the elements dX makes the set C_d if and only if $\left(X, \frac{n}{d}\right)_S \in P,\ 1 \le x \le \frac{n}{d}$. That is, $dX \in C_d$ if and only if X is in a minimal (P, S)-system $\left(\mathrm{mod}\,\frac{n}{d}\right)$. Replacing the particular (P, S)-system $\left(\mathrm{mod}\,\frac{n}{d}\right)$ by any arbitrary (P, S)-system $\left(\mathrm{mod}\,\frac{n}{d}\right)$ we get the theorem.

3.2. Theorem: $\sum_{d\in S_n\cap Q} \varphi_{P,S}\left(\frac{n}{d}\right) = n.$

Proof: Let C_d be as defined in the proof of Theorem 3.1. Then each C_d has $\varphi_{P,S}\left(\frac{n}{d}\right)$ elements; any two C_d's are disjoint and their union is the set $\{1,2,3,\ldots,n\}$

Hence $\sum_{d \in S_n \cap Q} \varphi_{P,S}\left(\frac{n}{d}\right) = n.$

3.3. Corollary: ([2], Theorem 5).

If P is a direct factor set and $\varphi_P(n)$ is the P-totient function of n then

$$\sum_{\substack{d \in Q \\ d | n}} \varphi_P\left(\frac{n}{d}\right) = n..$$

Proof: Taking $S_n = D_n$ in Theorem 3.2, we get corollary 3.3.

3.4. Corollary

If $\varphi_s(n)$ is the S-analogue of the Euler totient function

$$\sum_{d \in S_n} \varphi_S(d) = n.$$

Proof: Taking $P = \{1\}$ we find $Q = N$, the set of all natural numbers and $\varphi_{P,S}(n) = \varphi_S(n)$. Hence by Theorem 3.2, we get $\sum_{d \in S_n} \varphi_S\left(\frac{n}{d}\right) = n..$ Since $d \in S_n$ implies $\frac{n}{d} \in S_n$, the sum on the left is equal to $\sum_{d \in S_n} \varphi_S(n)$, proving the corollary.

3.5 Theorem: $\varphi_{P,S}(n) = \sum_{d \in S_n} d\, \mu_{P,S}\left(\frac{n}{d}\right)$

Proof: We have by Theorem 3.2 that

$$\sum_{d \in S_n \cap Q} \varphi_{P,S}\left(\frac{n}{d}\right) = N(n) \qquad \textbf{(3.6)}$$

Where $N(n) = n$ for all n.

Using (2.4) and (3.6) we have

$$\varphi_{P,S}(n) = \sum_{d \in S_n} N(d)\, \mu_{P,S}\left(\frac{n}{d}\right)$$

$$= \sum_{d \in S_n} d\, \mu_{P,S}\left(\frac{n}{d}\right),$$

Proving the theorem.

3.7 Theorem: $\varphi_{P,S}(n)=\sum_{d\in S_n\cap P}\varphi_S\left(\frac{n}{d}\right)$

Where $\varphi_S(n)$ is the S-analogue of the Euler function

Proof: By Theorem 3.5, (2.1) and (2.3)

$$\varphi_{P,S}(n)=\left(N\,\overline{S}\,\mu_{P,S}\right)(n)$$

$$=\left\{N\overline{S}\left(\mu_S\overline{S}\chi_P\right)\right\}(n)$$

$$=\left\{\left(N\,\overline{S}\mu_S\right)\overline{S}\chi_P\right\}(n)$$

$$=\left(\varphi_s\,\overline{S}\chi_P\right)(n),$$

Which gives the theorem.

3.8. Corollary: ([3], Theorem 8).

$$\varphi_P(n)=\sum_{\substack{d|n\\ d\in P}}\varphi\left(\frac{n}{d}\right)$$

Proof: In the case $S_n=D_n$ we have $\varphi_P(n)=\varphi(n)$, the Euler totient function. Now, the corollary follows from Theorem 3.7, taking $S_n=D_n$.

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